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ON CONSTITUTIVE EQUATIONS FOR ARBITRARY STRESS–STRAIN CONTROL IN MULTI-SURFACE PLASTICITY

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Abstract—The structure of incremental constitutive equations in multi-surface plasticity is discussed with respect to different choices of state and control variables. The state and control variables can combine stresses and strains as long as they are decomposed into energy-conjugate parts. A general uniqueness condition is established for non-associated flow rules and any choice of control variables. Furthermore, proper tangent constitutive matrices are given within each loading/unloading region. The theory is demonstrated in a simple example involving Tresca's yield condition.

1. INTRODUCTION

The structure of the incremental constitutive relations for elasto-plastic materials has been presented for a single smooth yield surface and different choices of state and control variables by Klisinski *et al.* (1992). In that paper it has been established that the choice of state space in which, for example, yield surfaces are described has no influence on the properties of the constitutive relations. However, the choice of the control variables has a major influence on these properties. In the present paper some of the concepts developed there are extended to multi-surface plasticity also known as corner or vertex plasticity where more than one yield surface can be loaded at the same time. The basic concepts for the flow theory of plasticity with non-smooth yield surfaces were introduced by Koiter (1953). The flow rules at a yield vertex have also been studied by Sewell (1974). The conditions for a unique response established there are, however, not necessary but only sufficient in a general case as shown later. Unconditional stability and uniqueness of incremental solutions have been described by Mroz (1963) and later Lubliner (1987) for generalized plasticity.

The use of mixed control, where both stress and strain components are combined, is necessary if the deformation process is subjected to some constraints. Typical examples are plane stress and plane strain conditions. However, they can just be treated as special cases of mixed control. One can easily imagine a situation where plane strain conditions are not exactly fulfilled, because a certain expansion is allowed. By using the general mixed control such cases can be calculated as easily as the plane strain condition. Another example, however, not covered in the present paper, is incompressibility imposed due to undrained conditions in soil mechanics as described by Runesson *et al.* (1991). From a conceptual point of view it seems always beneficial to introduce such constraints at the lowest possible level, i.e. in constitutive equations, and not at global level. Moreover, there is no doubt that the mixed control is the most natural in simulations of many experiments, as for example, the classical triaxial test. This is simply the same control as used in many experimental settings.

Matrix notation is used in the paper. Thus, second-order tensors are represented by column vectors, while fourth-order tensors are represented by square matrices. The scalar product of two vectors **a** and **b** is denoted by $\mathbf{a}^T\mathbf{b} = a_i b_i$ and the dyadic product by $\mathbf{a}\mathbf{b}^T = a_i b_j$, where superscript T denotes transpose.

2. BASIC FORMULATION

The basic ideas of the paper follow form Klisinski *et al.* (1992). The major difference is that the current formulation allows for more than one yield surface. This leads to some conceptual changes which will be pointed out in what follows. However, for details the reader should refer to the previous paper.

2.1. *Elastic–plastic state*

In multi-surface plasticity it is assumed that several yield surfaces exist limiting the elastic range to the set

$$B^{\#} = \{\boldsymbol{\sigma} : F_i^{\#}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_i) \leqslant 0\} \quad i = 1, 2, \ldots, n \tag{1}$$

in the stress space S, where $F_i^{\#}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_i)$ are the yield functions and $F_i^{\#}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_i) = 0$ represent the limit state for each function separately in the space $S \times K_i$. The internal parameter spaces K_i can be different for each yield function and the column vectors $\boldsymbol{\kappa}_i$ represent the current internal state used to model hardening/softening.

The elastic range eqn (1) can also be represented in the strain space E by an image of $B^{\#}$

$$B^* = \{\boldsymbol{\varepsilon} : F_i^*(\boldsymbol{\varepsilon}^e, \boldsymbol{\kappa}_i) \leqslant 0\}$$
$$F_i^*(\boldsymbol{\varepsilon}^e, \boldsymbol{\kappa}_i) \equiv F_i^{\#}(\mathbf{D}^e \boldsymbol{\varepsilon}^e, \boldsymbol{\kappa}_i) = F_i^{\#}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_i) \tag{2}$$

since the elastic strain $\boldsymbol{\varepsilon}^e$ is related to the stress $\boldsymbol{\sigma}$ by the relationship

$$\boldsymbol{\varepsilon}^e = \mathbf{C}^e \boldsymbol{\sigma}; \quad \boldsymbol{\sigma} = \mathbf{D}^e \boldsymbol{\varepsilon}^e; \quad \mathbf{D}^e = (\mathbf{C}^e)^{-1} \tag{3}$$

where $\mathbf{C}^e$ and $\mathbf{D}^e$ are the matrices of secant elastic compliance and stiffness moduli.

The yield surfaces and the elastic region can be represented in a variety of mixed stress–strain spaces as well. Stresses and strains can be decomposed into energy-conjugated parts

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2; \quad \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\varepsilon}_1 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\varepsilon}_2; \quad \boldsymbol{\sigma}_1 \cdot \boldsymbol{\varepsilon}_2 = \boldsymbol{\sigma}_2 \cdot \boldsymbol{\varepsilon}_1 = 0 \tag{4}$$

By a proper choice of coordinate system this decomposition can be expressed by

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1^T \quad \boldsymbol{\sigma}_2^T]^T; \quad \boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_1^T \quad \boldsymbol{\varepsilon}_2^T]^T \tag{5}$$

where only non-zero components are left in vectors $\boldsymbol{\sigma}_i$ and $\boldsymbol{\varepsilon}_i$.

The elastic laws

$$\begin{bmatrix} \boldsymbol{\varepsilon}_1^e \\ \boldsymbol{\varepsilon}_2^e \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11}^e & \mathbf{C}_{12}^e \\ \mathbf{C}_{21}^e & \mathbf{C}_{22}^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{bmatrix}; \quad \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11}^e & \mathbf{D}_{12}^e \\ \mathbf{D}_{21}^e & \mathbf{D}_{22}^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_1^e \\ \boldsymbol{\varepsilon}_2^e \end{bmatrix}; \quad \begin{matrix} \mathbf{C}_{21}^e = (\mathbf{C}_{12}^e)^T \\ \mathbf{D}_{21}^e = (\mathbf{D}_{12}^e)^T \end{matrix} \tag{6}$$

lead after partial inversion to the relation

$$\begin{bmatrix} \boldsymbol{\varepsilon}_1^e \\ \boldsymbol{\sigma}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11}^e & \mathbf{E}_{12}^e \\ \mathbf{E}_{21}^e & \mathbf{E}_{22}^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\varepsilon}_2^e \end{bmatrix} \qquad \begin{aligned} \mathbf{E}_{11}^e &= \mathbf{C}_{11}^e - \mathbf{C}_{12}^e(\mathbf{C}_{22}^e)^{-1}\mathbf{C}_{21}^e = (\mathbf{D}_{11}^e)^{-1} \\ \mathbf{E}_{12}^e &= \mathbf{C}_{12}^e(\mathbf{C}_{22}^e)^{-1} = -(\mathbf{D}_{11}^e)^{-1}\mathbf{D}_{12}^e \\ \mathbf{E}_{21}^e &= (\mathbf{C}_{22}^e)^{-1}\mathbf{C}_{21}^e = -\mathbf{D}_{21}^e(\mathbf{D}_{11}^e)^{-1} = -(\mathbf{E}_{12}^e)^T \\ \mathbf{E}_{22}^e &= (\mathbf{C}_{22}^e)^{-1} = \mathbf{D}_{22}^e - \mathbf{D}_{21}^e(\mathbf{D}_{11}^e)^{-1}\mathbf{D}_{12}^e \end{aligned} \tag{7}$$

where the constitutive matrix $\mathbf{E}$ is positive definite. The yield surfaces can be transformed to appropriate mixed spaces $S_1 \times E_2 \times K_i$ and the elastic region becomes

$$B = \{\boldsymbol{\sigma} : F_i(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i) \leqslant 0\} \tag{8}$$

where

$$F_i^{\#}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_i) = F_i^{\#}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\kappa}_i) = F_i^{\#}(\boldsymbol{\sigma}_1, \mathbf{E}_{21}^e\boldsymbol{\sigma}_1 + \mathbf{E}_{22}^e\boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i) \equiv F_i(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i)$$

$$F_i^{*}(\boldsymbol{\varepsilon}^e, \boldsymbol{\kappa}_i) = F_i^{*}(\boldsymbol{\varepsilon}_1^e, \boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i) = F_i^{*}(\mathbf{E}_{11}^e\boldsymbol{\sigma}_1 + \mathbf{E}_{12}^e\boldsymbol{\varepsilon}_2^e, \boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i) \equiv F_i(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2^e, \boldsymbol{\kappa}_i) \tag{9}$$

It is obvious that the yield surfaces can be defined in any space $S_1 \times E_2$ (assuming constant $\boldsymbol{\kappa}_i$) and transformed to any other mixed space as the transformations are unique. Therefore, the choice of space in which these surfaces are described is purely a matter of convenience.

2.2. *Plastic flow*

In plasticity the total strain rate is a sum of elastic and plastic rates

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \tag{10}$$

where from eqn (3) the elastic rate is related to the stress rate by

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{C}^e\dot{\boldsymbol{\sigma}}; \quad \dot{\boldsymbol{\sigma}} = \mathbf{D}^e\dot{\boldsymbol{\varepsilon}}^e; \tag{11}$$

Applying Koiter's rule the plastic strain rate is composed from components related to each yield surface

$$\dot{\boldsymbol{\varepsilon}}^p = \sum_i \dot{\boldsymbol{\varepsilon}}_i^p; \quad \dot{\boldsymbol{\varepsilon}}_i^p = \dot{\lambda}_i \mathbf{m}_i^{\#} \tag{12}$$

where for every yield surface the plastic strain rate is determined by the flow vector $\mathbf{m}_i^{\#}$, and where $\dot{\lambda}_i \geqslant 0$ is a scalar multiplier for yield surface i. In general $\mathbf{m}_i^{\#}$ are not proportional to the normal vectors $\mathbf{n}_i^{\#} = \mathbf{F}_{i,\sigma}^{\#}$ to each of the surfaces. So even if the most frequent choice is $\mathbf{m}_i^{\#} = \mathbf{n}_i^{\#}$, it is not presumed here, leading to a more general non-associated flow rule.

The hardening variables are functionals of plastic strain. The evolution rules are postulated in the form

$$\dot{\boldsymbol{\kappa}}_i = \mathbf{h}_i(\dot{\boldsymbol{\varepsilon}}_j^p) \tag{13}$$

where $\mathbf{h}_i$ are state-dependent, first order homogeneous functions of the plastic strain rates $\dot{\boldsymbol{\varepsilon}}_j^p$. However, the evolution of the internal state for each yield surface is in general coupled to the plastic strain rates generated by all other yield surfaces.

If the material state is such that $\mathbf{s} = (\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2) \in int(B)$ the response is always elastic, i.e. $\dot{\lambda}_i = 0$ for all yield surfaces, while when $\mathbf{s} = (\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2) \in \partial B$ the response can be either elastic or elasto-plastic depending on the loading/unloading criteria. If there exist at least one yield surface $F_i = 0$ for which $\dot{\lambda}_i > 0$ then plastic loading occurs and the state remains on the boundary of B. If, on the other hand, for all yield surfaces $\dot{\lambda}_i = 0$ and the state moves to the interior of B then unloading takes place. However, such conditions are insufficient because loading/unloading must be considered separately for each yield surface. Assuming that $F_i = 0$ the loading/unloading criteria for this yield surface are given by

$$\begin{array}{lll} \dot{\lambda}_i > 0 & \dot{F}_i = 0 & \text{plastic loading } (P_i) \\ \dot{\lambda}_i = 0 & \dot{F}_i = 0 & \text{neutral loading } (N_i) \\ \dot{\lambda}_i = 0 & \dot{F}_i < 0 & \text{elastic unloading } (E_i) \end{array} \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{unloading } (U_i) \tag{14}$$

and they must be exclusive and complete, i.e. always one and only one of them must be true. In what follows neutral loading will not be considered separately, but will always be included into the unloading, i.e. only two separate conditions for each yield surface will be distinguished (P_i) and (U_i). This should not cause any confusion, but it will considerably shorten the notation and derivations. The global criteria can be formulated as follows

$$\begin{aligned} &\exists i: F_i = 0 \quad P_i \quad \text{plastic loading } (P) \\ &\forall i: F_i = 0 \quad E_i \quad \text{elastic unloading } (E) \end{aligned} \tag{15}$$

whereas the neutral loading (N) occurs when neither P nor E is true.

3. CONSISTENCY CONDITION

3.1. *Stress space representation*

Assume that all yield surfaces are represented in stress space S and consider all who satisfy $F_i^{\#} = 0$. They will be called active yield surfaces.

3.1.1. *Stress control.* The consistency condition for each active yield surface is

$$\dot{F}_i^{\#} = \mathbf{n}_i^{\#} \cdot \dot{\boldsymbol{\sigma}} + F_{i,\kappa}^{\#} \cdot \dot{\boldsymbol{\kappa}}_i \leqslant 0; \quad \mathbf{n}_i^{\#} = F_{i,\sigma}^{\#} \tag{16}$$

From eqn (13)

$$\mathbf{F}_{i,\kappa}^{\#} \cdot \dot{\boldsymbol{\kappa}}_i = \sum_j \dot{\lambda}_j \mathbf{F}_{i,\kappa}^{\#} \cdot \mathbf{h}_i(\mathbf{m}_j^{\#}) \tag{17}$$

and arranging all scalar multipliers $\dot{\lambda}_j$ into one plastic multiplier vector $\dot{\boldsymbol{\lambda}}$ inequality eqn (16) can be represented as

$$\dot{F}_i^{\#} = \mathbf{n}_i^{\#} \cdot \dot{\boldsymbol{\sigma}} - \mathbf{H}_i \cdot \dot{\boldsymbol{\lambda}} \leqslant 0 \tag{18}$$

where

$$(\mathbf{H}_i)_j = -\mathbf{F}_{i,\kappa}^{\#} \cdot \mathbf{h}_i(\mathbf{m}_j^{\#}) \tag{19}$$

Furthermore, the consistency conditions eqn (18) for all active surfaces can be written as

$$\mathbf{N}^{\#} \dot{\boldsymbol{\sigma}} - \mathbf{H} \dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \tag{20}$$

where the rows of matrix $\mathbf{N}^{\#}$ are created by $(\mathbf{n}_i^{\#})^{\mathrm{T}}$ and the rows of matrix $\mathbf{H}$ by vectors $\mathbf{H}_i^{\mathrm{T}}$. Matrix $\mathbf{H}$ will be called the matrix of plastic moduli (under stress control). Before proceeding any further let us consider the equivalent consistency conditions for other types of control.

3.1.2. *Strain control.* Expressing $\dot{\boldsymbol{\sigma}}$ via $\dot{\boldsymbol{\varepsilon}}$

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^{\mathrm{e}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \mathbf{D}^{\mathrm{e}}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) = \mathbf{D}^{\mathrm{e}}\left(\dot{\boldsymbol{\varepsilon}} - \sum_i \dot{\lambda}_i \mathbf{m}_i^{\#}\right) = \mathbf{D}^{\mathrm{e}}(\dot{\boldsymbol{\varepsilon}} - \mathbf{M}^{\#} \dot{\boldsymbol{\lambda}}) \tag{21}$$

where $\mathbf{M}^{\#}$ is created by arranging $\mathbf{m}_i^{\#}$ column by column, the consistency condition eqn (20) can be stated as

$$\begin{aligned} &\mathbf{N}^* \dot{\boldsymbol{\varepsilon}} - \mathbf{K}^* \dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \\ &\mathbf{N}^* = \mathbf{N}^{\#} \mathbf{D}^{\mathrm{e}}; \quad \mathbf{K}^* = \mathbf{H} + \mathbf{N}^{\#} \mathbf{D}^{\mathrm{e}} \mathbf{M}^{\#} \end{aligned} \tag{22}$$

where $\mathbf{N}^*$ represents the transformed normal vectors and $\mathbf{K}^*$ is the matrix of plastic moduli under strain control. Notice that condition (22) is of the same type as eqn (20) but with different matrices and control. If only associated flow rules are applied then $\mathbf{M}^{\#} = (\mathbf{N}^{\#})^{\mathrm{T}}$.

3.1.3. *Mixed control.* Assume that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\varepsilon}_2$ are the control variables and $\dot{\mathbf{c}} = [\dot{\boldsymbol{\sigma}}_1^{\mathrm{T}} \quad \dot{\boldsymbol{\varepsilon}}_2^{\mathrm{T}}]^{\mathrm{T}}$ denotes their increment. The remaining part of the stress increment $\dot{\boldsymbol{\sigma}}_2$ can be expressed via $\dot{\mathbf{c}}$ as

$$\dot{\boldsymbol{\sigma}}_2 = \mathbf{E}^{\mathrm{e}}_{21}\dot{\boldsymbol{\sigma}}_1 + \mathbf{E}^{\mathrm{e}}_{22}(\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_2) = \mathbf{E}^{\mathrm{e}}_{21}\dot{\boldsymbol{\sigma}}_1 + \mathbf{E}^{\mathrm{e}}_{22}(\dot{\boldsymbol{\varepsilon}}_2 - \mathbf{M}^{\#}_2\dot{\boldsymbol{\lambda}}) \tag{23}$$

where the notation is fairly obvious and follows from the decomposition, for example

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{1i} = \dot{\lambda}_i \mathbf{m}^{\#}_{1i}\,; \quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{2i} = \dot{\lambda}_i \mathbf{m}^{\#}_{2i} \tag{24}$$

and matrix $\mathbf{M}^{\#}_2$ is created by arranging vectors $\mathbf{m}^{\#}_{2i}$ column by column. The consistency condition can be rewritten as follows

$$\mathbf{N}^{\#}\dot{\boldsymbol{\sigma}} - \mathbf{H}\dot{\boldsymbol{\lambda}} = [\mathbf{N}^{\#}_1\mathbf{N}^{\#}_2]\begin{bmatrix}\dot{\boldsymbol{\sigma}}_1\\ \dot{\boldsymbol{\sigma}}_2\end{bmatrix} - \mathbf{H}\dot{\boldsymbol{\lambda}} = [\mathbf{N}^{\#}_1 + \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{21}, \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{22}]\begin{bmatrix}\dot{\boldsymbol{\sigma}}_1\\ \dot{\boldsymbol{\varepsilon}}_2\end{bmatrix} - (\mathbf{H} + \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{22}\mathbf{M}^{\#}_2)\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \tag{25}$$

and introducing

$$\mathbf{N}_1 = \mathbf{N}^{\#}_1 + \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{21}\,; \quad \mathbf{N}_2 = \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{22}\,; \quad \mathbf{N} = [\mathbf{N}_1, \mathbf{N}_2]\,; \quad \mathbf{K} = \mathbf{H} + \mathbf{N}^{\#}_2\mathbf{E}^{\mathrm{e}}_{22}\mathbf{M}^{\#}_2 \tag{26}$$

simply as

$$\mathbf{N}\dot{\mathbf{c}} - \mathbf{K}\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \tag{27}$$

where $\mathbf{N}$, according to eqn (26), represents the transformed normal vectors and $\mathbf{K}$ is the matrix of generalized plastic moduli under mixed control.

3.2. *Strain space representation*

Assume that all yield surfaces are represented in strain space E and only consider the active surfaces, $F^*_i = 0$.

3.2.1. *Strain control.* The consistency condition can be written as

$$\dot{F}^*_i = \mathbf{n}^*_i \cdot \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \mathbf{F}^*_{i,\kappa} \cdot \dot{\boldsymbol{\kappa}}_i \leqslant 0\,; \quad \mathbf{n}^*_i = \mathbf{F}^*_{i,\varepsilon^{\mathrm{e}}} \tag{28}$$

From eqn (2) it follows that

$$\mathbf{n}^*_i = \mathbf{D}^{\mathrm{e}}\mathbf{n}^{\#}_i\,; \quad \mathbf{F}^*_{i,\kappa} = \mathbf{F}^{\#}_{i,\kappa} \tag{29}$$

and introducing the transformed plastic flow vectors in a way similar to eqn (29)

$$\mathbf{m}^*_i = \mathbf{D}^{\mathrm{e}}\mathbf{m}^{\#}_i\,; \quad \mathbf{m}^{\#}_i = \mathbf{C}^{\mathrm{e}}\mathbf{m}^*_i; \quad \mathbf{M}^* = \mathbf{D}^{\mathrm{e}}\mathbf{M}^{\#}\,; \quad \mathbf{M}^{\#} = \mathbf{C}^{\mathrm{e}}\mathbf{M}^* \tag{30}$$

the consistency conditions (28) can be represented by

$$\mathbf{N}^*\dot{\boldsymbol{\varepsilon}} - \mathbf{K}^*\dot{\boldsymbol{\lambda}} \leqslant 0 \tag{31}$$

where the rows of $\mathbf{N}^*$ are created by $(\mathbf{n}^*_i)^{\mathrm{T}}$ whereas

$$\mathbf{K}^* = \mathbf{H} + \mathbf{N}^*\mathbf{M}^{\#} = \mathbf{H} + \mathbf{N}^*\mathbf{C}^{\mathrm{e}}\mathbf{M}^* \tag{32}$$

is the same matrix as in eqn (22).

3.2.2. *Stress control.* Expressing $\dot{\boldsymbol{\varepsilon}}$ in terms of $\dot{\boldsymbol{\sigma}}$

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}^e\dot{\boldsymbol{\sigma}}+\dot{\boldsymbol{\varepsilon}}^p = \mathbf{C}^e(\dot{\boldsymbol{\sigma}}+\mathbf{M}^*\dot{\boldsymbol{\lambda}}) \tag{33}$$

and substituting into eqn (31) one obtains

$$\mathbf{N}^*\dot{\boldsymbol{\varepsilon}}-\mathbf{K}^*\dot{\boldsymbol{\lambda}} = \mathbf{N}^\#\dot{\boldsymbol{\sigma}}-\mathbf{K}^\#\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}$$

$$\mathbf{N}^\# = \mathbf{N}^*\mathbf{C}^e; \quad \mathbf{K}^\# = \mathbf{K}^*-\mathbf{N}^*\mathbf{C}^e\mathbf{M}^* = \mathbf{H} \tag{34}$$

3.2.3. *Mixed control.* Using, as before, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\varepsilon}_2$ as control variables the increment of $\boldsymbol{\varepsilon}_1$ can be expressed as

$$\dot{\boldsymbol{\varepsilon}}_1 = \mathbf{E}^e_{11}\dot{\boldsymbol{\sigma}}_1+\mathbf{E}^e_{12}(\dot{\boldsymbol{\varepsilon}}_2-\dot{\boldsymbol{\varepsilon}}^p_2)+\dot{\boldsymbol{\varepsilon}}^p_1 = \mathbf{E}^e_{11}\dot{\boldsymbol{\sigma}}_1+\mathbf{E}^e_{12}(\dot{\boldsymbol{\varepsilon}}_2-\mathbf{M}^\#_2\dot{\boldsymbol{\lambda}})+\mathbf{M}^\#_1\dot{\boldsymbol{\lambda}} \tag{35}$$

Condition (31) can be written as

$$\mathbf{N}\dot{\mathbf{c}}-\mathbf{K}\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \tag{36}$$

where

$$\mathbf{N}_1 = \mathbf{N}^*_1\mathbf{E}^e_{11}; \quad \mathbf{N}_2 = \mathbf{N}^*_2+\mathbf{N}^*_1\mathbf{E}^e_{12}; \quad \mathbf{N} = [\mathbf{N}_1, \mathbf{N}_2]; \quad \mathbf{K} = \mathbf{K}^*-\mathbf{N}^*_1\mathbf{E}^e_{11}\mathbf{M}^*_1 \tag{37}$$

Once again the same matrices as in eqn (27) appear.

3.3. *Mixed space representation—mixed control*

Let us also consider a special case of mixed space representation in which both the state and control variables are the same, i.e. $\mathbf{c} = \mathbf{s}$. Denoting

$$\mathbf{n}_{i1} = \mathbf{F}_{i,\sigma_1}; \quad \mathbf{n}_{i2} = \mathbf{F}_{i,\varepsilon^e_2} \tag{38}$$

the appropriate gradient transformations according to eqn (9) are

$$\begin{aligned} \mathbf{n}_{i1} &= \mathbf{n}^\#_{i1}-\mathbf{E}^e_{12}\mathbf{n}^\#_{i2} = \mathbf{E}^e_{12}\mathbf{n}^*_{i1}; \\ \mathbf{n}_{i2} &= \mathbf{E}^e_{22}\mathbf{n}^\#_{i2} = -\mathbf{E}^e_{21}\mathbf{n}^*_{i1}+\mathbf{n}^*_{i2}; \end{aligned} \qquad \mathbf{F}_{i,\kappa} = \mathbf{F}^\#_{i,\kappa} = \mathbf{F}^*_{i,\kappa} \tag{39}$$

The same transformations as for $\mathbf{n}_{i1}$ and $\mathbf{n}_{i2}$ hold for $\mathbf{m}_{i1}$ and $\mathbf{m}_{i2}$.

The consistency condition is expressed as

$$\mathbf{N}\dot{\mathbf{c}}-\mathbf{K}\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0} \tag{40}$$

where matrices $\mathbf{N}$ and $\mathbf{M}$ are obtained from vectors $\mathbf{n}_i$ and $\mathbf{m}_i$ in the same way as before and

$$\mathbf{N} = [\mathbf{N}_1, \quad \mathbf{N}_2]; \qquad \begin{aligned} \mathbf{N}_1 &= \mathbf{N}^\#_1+\mathbf{N}^\#_2\mathbf{E}^e_{21} = \mathbf{N}^*_1\mathbf{E}^e_{11} \\ \mathbf{N}_2 &= \mathbf{N}^\#_2\mathbf{E}^e_{22} = \mathbf{N}^*_2+\mathbf{N}^*_1\mathbf{E}^e_{12} \end{aligned}$$

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_1 \\ \mathbf{M}_2\end{bmatrix}; \qquad \begin{aligned} \mathbf{M}_1 &= \mathbf{M}^\#_1-\mathbf{E}^e_{12}\mathbf{M}^\#_2 = \mathbf{E}^e_{11}\mathbf{M}^*_1 \\ \mathbf{M}_2 &= \mathbf{E}^e_{22}\mathbf{M}^\#_2 = \mathbf{M}^*_2-\mathbf{E}^e_{21}\mathbf{M}^*_1 \end{aligned}$$

$$\mathbf{K} = \mathbf{H}+\mathbf{N}_2\mathbf{C}^e_{22}\mathbf{M}_2 = \mathbf{H}+\mathbf{N}^\#_2\mathbf{E}^e_{22}\mathbf{M}^\#_2 = \mathbf{K}^*-\mathbf{N}^*_1\mathbf{E}^e_{11}\mathbf{M}^*_1 \tag{41}$$

Notice that in all considered cases the consistency condition is of the type (40) and the form of matrices $\mathbf{N}$ and $\mathbf{K}$ only depends on the control variable $\mathbf{c}$ and not the state variable $\mathbf{s}$. Therefore, the space used to describe the yield surfaces does not matter, whereas the control is of primary importance. The next part of the paper is dedicated to studies of the system of inequalities eqn (40).

4. UNIQUENESS OF RESPONSE

Consider the uniqueness requirements if n yield surfaces are active, i.e. $F_i = 0$ for $i = 1, 2, \ldots, n$, and the consistency condition is of its general form eqn (40). Unloading occurs when $\dot{\boldsymbol{\lambda}} = \mathbf{0}$ and then the condition is simply

$$\mathbf{N}\dot{\mathbf{c}} \leqslant \mathbf{0} \tag{42}$$

Let us require that the above set is not degenerate in the control space, i.e. it does not diminish to a point, line, etc., because otherwise the elastic region would have a rather strange shape. Let us also assume that all conditions in eqn (42) are needed to describe this set, i.e. the fact that some of the inequalities in eqn (42) are satisfied does not inevitably imply that some other inequality in eqn (42) is identically fulfilled.

Next consider loading of just one surface, i.e. the loading multiplier vector is of the form $\dot{\boldsymbol{\lambda}} = [\dot{\lambda}_i \quad \mathbf{0}]^{\mathrm{T}}$ where $\dot{\lambda}_i > 0$, obtained by a proper reordering of the inequalities. The consistency condition becomes

$$\begin{bmatrix} \mathbf{n}_i^{\mathrm{T}} \\ \mathbf{N}_0 \end{bmatrix} \dot{\mathbf{c}} - \begin{bmatrix} K_{ii} & \mathbf{k}_{i0} \\ \mathbf{k}_{0i} & \mathbf{K}_{00} \end{bmatrix} \begin{bmatrix} \dot{\lambda}_i \\ \mathbf{0} \end{bmatrix} \begin{matrix} = \\ \leqslant \end{matrix} \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \mathbf{n}_i^{\mathrm{T}}\dot{\mathbf{c}} - K_{ii}\dot{\lambda}_i = 0 \\ \mathbf{N}_0\dot{\mathbf{c}} - \mathbf{k}_{0i}\dot{\lambda} \leqslant \mathbf{0} \end{matrix} \tag{43}$$

providing the solution

$$\dot{\lambda}_i = \frac{1}{K_{ii}} \mathbf{n}_i^{\mathrm{T}}\dot{\mathbf{c}} > 0\,; \quad (\mathbf{N}_0 - \mathbf{k}_{0i}K_{ii}^{-1}\mathbf{n}_i^{\mathrm{T}})\dot{\mathbf{c}} \leqslant \mathbf{0} \tag{44}$$

The first inequality can only be distinguished from the corresponding condition in eqn (42), i.e. $\mathbf{n}_i^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0$, if $K_{ii} > 0$ and then the solution (44) is also unique. When the above reasoning is repeated for all yield surfaces it must be required that $K_{ii} > 0$ for $i = 1, 2, \ldots, n$.

The next step is to consider a transition between two loading cases defined by the following plastic multiplier vectors

$$\dot{\boldsymbol{\lambda}} = [\dot{\boldsymbol{\lambda}}_+^{\mathrm{T}} \quad 0 \quad \mathbf{0}]^{\mathrm{T}} \quad \text{and} \quad \dot{\boldsymbol{\lambda}} = [\dot{\boldsymbol{\lambda}}_+^{\mathrm{T}} \quad \dot{\lambda}_i \quad \mathbf{0}]^{\mathrm{T}}\,; \quad \dot{\boldsymbol{\lambda}}_+ > \mathbf{0}\,; \quad \dot{\lambda}_i > 0 \tag{45}$$

The first vector leads to the following consistency condition

$$\begin{bmatrix} \mathbf{N}_+ \\ \mathbf{n}_i^{\mathrm{T}} \\ \mathbf{N}_0 \end{bmatrix} \dot{\mathbf{c}} - \begin{bmatrix} \mathbf{K}_+ & \mathbf{k}_{+i} & \mathbf{K}_{+0} \\ \mathbf{k}_{i+} & K_{ii} & \mathbf{k}_{i0} \\ \mathbf{K}_{0+} & \mathbf{k}_{0i} & \mathbf{K}_{00} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\lambda}}_+ \\ 0 \\ \mathbf{0} \end{bmatrix} \begin{matrix} = \\ \leqslant \\ \leqslant \end{matrix} \begin{bmatrix} \mathbf{0} \\ 0 \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \mathbf{N}_+\dot{\mathbf{c}} - \mathbf{K}_+\dot{\boldsymbol{\lambda}}_+ = \mathbf{0} \\ \begin{bmatrix} \mathbf{n}_i^{\mathrm{T}} \\ \mathbf{N}_0 \end{bmatrix} \dot{\mathbf{c}} - \begin{bmatrix} \mathbf{k}_{i+} \\ \mathbf{K}_{0+} \end{bmatrix} \dot{\boldsymbol{\lambda}}_+ \leqslant \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix} \end{matrix} \tag{46}$$

providing the solution

$$\dot{\boldsymbol{\lambda}}_+ = \mathbf{K}_+^{-1}\mathbf{N}_+\dot{\mathbf{c}} > \mathbf{0}\,; \quad \left(\begin{bmatrix} \mathbf{n}_i^{\mathrm{T}} \\ \mathbf{N}_0 \end{bmatrix} - \begin{bmatrix} \mathbf{k}_{i+} \\ \mathbf{K}_{0+} \end{bmatrix} \mathbf{K}_+^{-1}\mathbf{n}_+ \right) \dot{\mathbf{c}} \leqslant \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix} \tag{47}$$

and the condition connected with yield surface $F_i = 0$ is

$$(\mathbf{n}_i^{\mathrm{T}} - \mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{N}_+)\dot{\mathbf{c}} \leqslant 0 \tag{48}$$

The second plastic multiplier vector gives

$$\begin{bmatrix}\mathbf{N}_+\\ \mathbf{n}_i^{\mathrm{T}}\\ \mathbf{N}_0\end{bmatrix}\dot{\mathbf{c}}-\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i} & \mathbf{K}_{+0}\\ \mathbf{k}_{i+} & K_{ii} & \mathbf{k}_{i0}\\ \mathbf{K}_{0+} & \mathbf{k}_{0i} & \mathbf{K}_{00}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\lambda}}_+\\ \dot{\lambda}_i\\ \mathbf{0}\end{bmatrix}\begin{matrix}=\\=\\ \leqslant\end{matrix}\begin{bmatrix}\mathbf{0}\\0\\ \mathbf{0}\end{bmatrix} \quad \text{or} \quad \begin{matrix}\begin{bmatrix}\mathbf{N}_+\\ \mathbf{n}_i^{\mathrm{T}}\end{bmatrix}\dot{\mathbf{c}}-\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\lambda}}_+\\ \dot{\lambda}_i\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\0\end{bmatrix}\\ \mathbf{N}_0\dot{\mathbf{c}}-[\mathbf{K}_{0+}\mathbf{k}_{0i}]\begin{bmatrix}\dot{\boldsymbol{\lambda}}_+\\ \dot{\lambda}_i\end{bmatrix}\leqslant\mathbf{0}\end{matrix} \tag{49}$$

and the solution turns out to be

$$\begin{bmatrix}\dot{\boldsymbol{\lambda}}_+\\ \dot{\lambda}_i\end{bmatrix}=\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{N}_+\\ \mathbf{n}_i^{\mathrm{T}}\end{bmatrix}\dot{\mathbf{c}}>\mathbf{0}; \quad \left(\mathbf{N}_0-[\mathbf{K}_{0+}\mathbf{k}_{0i}]\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{N}_+\\ \mathbf{n}_i^{\mathrm{T}}\end{bmatrix}\right)\dot{\mathbf{c}}\leqslant\mathbf{0} \tag{50}$$

where the inverse matrix can be represented by

$$\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix}^{-1}=\frac{1}{K_{ii}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{k}_{+i}}\begin{bmatrix}[(K_{ii}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{k}_{+i})\mathbf{I}+\mathbf{K}_+^{-1}\mathbf{k}_{+i}\mathbf{k}_{i+}]\mathbf{K}_+^{-1} & -\mathbf{K}_+^{-1}\mathbf{k}_{+i}\\ -\mathbf{k}_{i+}\mathbf{K}_+^{-1} & 1\end{bmatrix} \tag{51}$$

The most important condition is the one that involves $\dot{\lambda}_i$

$$\dot{\lambda}_i=\frac{1}{K_{ii}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{k}_{+i}}(\mathbf{n}_i^{\mathrm{T}}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{N}_+)\dot{\mathbf{c}}>0 \tag{52}$$

and it must be compared with eqn (48). These two conditions describe separate half-spaces with a common boundary and to be able to distinguish between them it must be required that

$$K_{ii}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{k}_{-i}>0 \tag{53}$$

This condition can be expressed in terms of the determinant of an appropriate matrix. First notice that the inverse of a matrix $\mathbf{K}_+$ can be represented as

$$\mathbf{K}_+^{-1}=\frac{1}{\det(\mathbf{K}_+)}\operatorname{adj}(\mathbf{K}_+) \tag{54}$$

where $\operatorname{adj}(\mathbf{K}_+)$ is the adjoint matrix to $\mathbf{K}_+$. Substituting into eqn (53) and assuming that $\det(\mathbf{K}_+)>0$ it is possible to rewrite this condition as

$$\begin{gathered}K_{ii}-\mathbf{k}_{i+}\mathbf{K}_+^{-1}\mathbf{k}_{+i}=K_{ii}-\frac{1}{\det(\mathbf{K}_+)}\mathbf{k}_{i+}\operatorname{adj}(\mathbf{K}_+)\mathbf{k}_{+i}>0\\ \det(\mathbf{K}_+)>0\Rightarrow K_{ii}\det(\mathbf{K}_+)-\mathbf{k}_{i+}\operatorname{adj}(\mathbf{K}_+)\mathbf{k}_{+i}>0\end{gathered} \tag{55}$$

The last inequality is equivalent to the requirement that the determinant of the following sub-matrix is positive

$$\det\begin{bmatrix}\mathbf{K}_{+} & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix} > 0 \tag{56}$$

The above reasoning must be repeated to cover all possible combinations of loading/unloading for all active surfaces. Mathematical induction assures that the above condition is independent of the number of active surfaces. Notice also that if condition (56) is satisfied then the solution (50) is unique. Therefore, both the transitions between different loading regions and the solutions will be unique if and only if eqn (56) holds in all cases.

Summarizing, the sufficient and necessary condition for uniqueness of the response under given control can be described as follows:

All principle minors of all orders of the matrix of generalized plastic moduli must be positive. (57)

A principal minor of order k is a determinant of a principal submatrix whose entries are located in a given $n \times n$ matrix at the intersection of k distinct columns and k distinct rows who cross on the main diagonal i.e. have the same indices. In particular principal minors of order 1 are just the main diagonal entries, whereas there is only one principal minor of order n and it is the determinant of the entire matrix.

For example, if $\mathbf{K}$ is a 3×3 matrix then the above condition (57) can be written as

$$\begin{matrix} K_{11} > 0 \\ K_{22} > 0 \\ K_{33} > 0 \end{matrix} \quad \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} > 0 \quad \begin{vmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{vmatrix} > 0 \quad \begin{vmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{vmatrix} > 0 \quad \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} > 0 \tag{58}$$

Notice that the derived uniqueness condition is not equivalent to the requirement that matrix $\mathbf{K}$ must be positive definite as suggested by Sewell (1974). For example, matrix

$$\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$$

satisfies (57), but is not positive definite since

$$[1 \quad 1]\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 < 0.$$

On the other hand if matrix $\mathbf{K}$ is positive definite then it also satisfies the uniqueness condition. The proof is based on the fact that any principal submatrix of such a matrix must also be positive definite. It can easily be shown by choosing proper zero entries in vector $\mathbf{x}$ appearing in the definition of a positive definite matrix, $\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} > 0$. Furthermore, the determinant of any positive definite matrix is positive (see appendix). Therefore, the uniqueness condition is satisfied and the requirement that $\mathbf{K}$ is positive definite is sufficient but not necessary to achieve uniqueness of response.

However, both considered conditions are equivalent for symmetric matrices $\mathbf{K}$. Moreover, for symmetric matrices it is not necessary to check all the principal minors, but only the corner minors, i.e. determinants of submatrices consisting of the first k rows and columns.

4.1. *Special cases*

It may also be interesting to take a look at some special cases, because in the general considerations it has been required that each case is distinct and the transition between them is continuous. But sometimes it is know "*a priori*" that it is possible to distinguish between fewer regions than all previously considered. One such case occurs when two yield surfaces are tangential to each other. Another example involves three yield lines crossing at the same point on a plane.

4.1.1. *Two tangential yield surfaces.* Assume that the normal vectors to two yield surfaces are the same, i.e. $\mathbf{n}_2 = \mathbf{n}_1$. Notice that it is sufficient to assume that they are just proportional with a positive proportionality coefficient as all equations can be scaled, but let us neglect this operation for the sake of simplicity. Consider the following four loading/unloading combinations

$$
\begin{aligned}
&(U) \quad \dot{\lambda}_1 = \dot{\lambda}_2 = 0 \Rightarrow \mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \\
&(P_1) \quad \dot{\lambda}_1 > 0 \wedge \dot{\lambda}_2 = 0 \Rightarrow \frac{1}{K_{11}}\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} > 0; \quad \left(1-\frac{K_{21}}{K_{11}}\right)\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \\
&(P_2) \quad \dot{\lambda}_1 = 0 \wedge \dot{\lambda}_2 > 0 \Rightarrow \frac{1}{K_{22}}\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} > 0; \quad \left(1-\frac{K_{12}}{K_{22}}\right)\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \\
&(P_{12}) \quad \dot{\lambda}_1 > 0 \wedge \dot{\lambda}_2 > 0 \Rightarrow \frac{K_{22}-K_{12}}{\Delta}\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} > 0; \quad \frac{K_{11}-K_{21}}{\Delta}\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} > 0
\end{aligned}
\tag{59}
$$

where $\Delta = K_{11}K_{22} - K_{12}K_{21}$.

Notice that it is impossible to distinguish between all four cases, because the vector product $\mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}}$ appearing in eqn (59) can only have two different signs. The problem is when the two potential loading/unloading conditions are consistent. The unloading condition is clear and, for a given matrix $\mathbf{K}$, only one of the other cases can occur. However, at the same time the remaining loading cases must be false since they cannot happen. If $\Delta > 0$ the three loading cases give

$$
\begin{aligned}
&(P_1) \quad 0 < K_{11} \leqslant K_{21} \\
&(P_2) \quad 0 < K_{22} \leqslant K_{12} \\
&(P_{12}) \quad K_{12} < K_{22}; \quad K_{21} < K_{11}
\end{aligned}
\tag{60}
$$

and loading of two surfaces (P_{12}) is always exclusive with loading of just one surface. However, it seems that the first two conditions can be true at the same time, but then $\Delta < 0$ in spite of the initial assumption. Therefore, all loading cases are exclusive and only one of them happens. But, if for example loading of the first surface occurs it is no longer necessary that $K_{22} > 0$. As a result the general uniqueness condition need not to be satisfied. If $\Delta < 0$ then the loading conditions are

$$
\begin{aligned}
&(P_1) \quad 0 < K_{11} \leqslant K_{21} \\
&(P_2) \quad 0 < K_{22} \leqslant K_{12} \\
&(P_{12}) \quad K_{22} < K_{12}; \quad K_{11} < K_{21}
\end{aligned}
\tag{61}
$$

The consecutive cases can be distinguished from the other two if

$$
\begin{aligned}
&(P_1) \quad 0 < K_{11} \leqslant K_{21}; \quad K_{12} < K_{22} \\
&(P_2) \quad 0 < K_{22} \leqslant K_{12}; \quad K_{21} \leqslant K_{11} \\
&(P_{12}) \quad K_{22} < K_{12}; \quad K_{22} \leqslant 0; \quad K_{11} < K_{21}; \quad K_{11} \leqslant 0
\end{aligned}
\tag{62}
$$

It should be noticed that these conditions are rather restrictive. Moreover, the regions described by them are not connected. For example, the transition between (P_1) and (P_{12}) requires that

$$K_{12} = K_{22} \leqslant 0; \quad 0 = K_{11} \leqslant K_{21} \tag{63}$$

but then $\Delta = -K_{22}K_{21} > 0$, i.e. such a boundary does not exist. On the other hand, if the general uniqueness conditions (57) are satisfied then the response is also unique in this special case. The reasoning presented in the previous section remains valid resulting in sufficient but not longer necessary conditions.

4.1.2. *Three yield lines in a plane.* Let us consider a 2-dimensional case with three separate yield lines crossing at the same point. Thus if the elastic unloading region is not degenerate, i.e. it is not represented by the intersection point alone then only two out of three inequalities in eqn (42) are required to describe this region. To simplify further our considerations assume

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{n}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{64}$$

It is clear that

$$\forall \dot{\mathbf{c}} \quad \begin{matrix} \mathbf{n}_1^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \\ \mathbf{n}_2^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \end{matrix} \Rightarrow \mathbf{n}_3^{\mathrm{T}}\dot{\mathbf{c}} \leqslant 0 \tag{65}$$

so the unloading region can be defined by considering the first two conditions only. The consistency condition can be expressed as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \dot{\mathbf{c}} - \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} \leqslant \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{66}$$

and there are 8 different loading/unloading cases defined by

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \dot{\lambda}_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \dot{\lambda}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \dot{\lambda}_3 \end{bmatrix}, \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix}, \begin{bmatrix} \dot{\lambda}_1 \\ 0 \\ \dot{\lambda}_3 \end{bmatrix}, \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} \tag{67}$$

One can easily notice that analytical considerations of all of them may be very laborious. Therefore, numerical testing has been performed instead. The results indicate that there exist unique cases outside the region described by the general criterion. It seems that an effort to establish criteria that take care of these incidental regions of uniqueness has no practical significance. Once again the uniqueness conditions (58) are sufficient but not necessary in this special case.

5. RESPONSE

Now when the necessary (or just sufficient in some special cases) uniqueness conditions have been established it is vital to consider the response of an elastic–plastic model. As before only the active yield surfaces are considered, i.e. such that $F_i = 0$ for $i = 1, 2, \ldots, n$.

5.1. *Stress control*

It is assumed that the matrix of plastic moduli $\mathbf{H}$ satisfies the uniqueness condition (57). If unloading occurs the response is purely elastic, i.e.

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}^{e}\dot{\boldsymbol{\sigma}} \quad \text{if} \quad \mathbf{N}^{\#}\dot{\boldsymbol{\sigma}} \leqslant \mathbf{0} \tag{68}$$

Otherwise, all different loading conditions must be considered. Let us decompose all matrices into parts connected with these scalar multipliers which are positive and zero, respectively. The consistency conditions may be written

$$\begin{bmatrix} \mathbf{N}^{\#}_{+} \\ \mathbf{N}^{\#}_{0} \end{bmatrix} \dot{\boldsymbol{\sigma}} - \begin{bmatrix} \mathbf{H}_{+} & \mathbf{H}_{+0} \\ \mathbf{H}_{0+} & \mathbf{H}_{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\lambda}}_{+} \\ \mathbf{0} \end{bmatrix} \begin{matrix} = \\ \leqslant \end{matrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \mathbf{N}^{\#}_{+}\dot{\boldsymbol{\sigma}} - \mathbf{H}_{+}\dot{\boldsymbol{\lambda}}_{+} = \mathbf{0} \\ \mathbf{N}^{\#}_{0}\dot{\boldsymbol{\sigma}} - \mathbf{H}_{0+}\dot{\boldsymbol{\lambda}}_{+} \leqslant \mathbf{0} \end{matrix} \tag{69}$$

and the corresponding region where they are satisfied is defined by

$$\dot{\boldsymbol{\lambda}}_{+} = \mathbf{H}_{+}^{-1}\mathbf{N}^{\#}_{+}\dot{\boldsymbol{\sigma}} > \mathbf{0}; \quad (\mathbf{N}^{\#}_{0} - \mathbf{H}_{0+}\mathbf{H}_{+}^{-1}\mathbf{N}^{\#}_{+})\dot{\boldsymbol{\sigma}} \leqslant \mathbf{0} \tag{70}$$

The plastic strain increment is represented by

$$\dot{\boldsymbol{\varepsilon}}^{p} = \mathbf{M}^{\#}\dot{\boldsymbol{\lambda}} = \mathbf{M}^{\#}_{+}\dot{\boldsymbol{\lambda}}_{+} = \mathbf{M}^{\#}_{+}\mathbf{H}_{+}^{-1}\mathbf{N}^{\#}_{+}\dot{\boldsymbol{\sigma}} \tag{71}$$

whereas the total strain increment can be calculated from the tangent compliance relation

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}\dot{\boldsymbol{\sigma}}; \quad \mathbf{C} = \mathbf{C}^{e} + \mathbf{C}^{p}; \quad \mathbf{C}^{p} = \mathbf{M}^{\#}_{+}\mathbf{H}_{+}^{-1}\mathbf{N}^{\#}_{+} \tag{72}$$

in the region defined by eqn (70). There exist as many forms of the plastic compliance matrix as there are regions and choices of $\dot{\boldsymbol{\lambda}}_{+} > 0$. For n active yield surfaces there are $2^{n} - 1$ such matrices.

5.2. *Strain control*

It is assumed that right now the matrix of plastic moduli under strain control $\mathbf{K}^{*}$ satisfies the uniqueness condition (57). Under unloading the response is elastic

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^{e}\dot{\boldsymbol{\varepsilon}} \quad \text{if} \quad \mathbf{N}^{*}\dot{\boldsymbol{\varepsilon}} \leqslant \mathbf{0} \tag{73}$$

Otherwise loading of at least one yield surface occurs and then the general consistency conditions can be written

$$\begin{bmatrix} \mathbf{N}^{*}_{+} \\ \mathbf{N}^{*}_{0} \end{bmatrix} \dot{\boldsymbol{\varepsilon}} - \begin{bmatrix} \mathbf{K}^{*}_{+} & \mathbf{K}^{*}_{+0} \\ \mathbf{K}^{*}_{0+} & \mathbf{K}^{*}_{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\lambda}}_{+} \\ \mathbf{0} \end{bmatrix} \begin{matrix} = \\ \leqslant \end{matrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \mathbf{N}^{*}_{+}\dot{\boldsymbol{\varepsilon}} - \mathbf{K}^{*}_{+}\dot{\boldsymbol{\lambda}}_{+} = \mathbf{0} \\ \mathbf{N}^{*}_{0}\dot{\boldsymbol{\varepsilon}} - \mathbf{K}^{*}_{0+}\dot{\boldsymbol{\lambda}}_{+} \leqslant \mathbf{0} \end{matrix} \tag{74}$$

providing the solution

$$\dot{\boldsymbol{\lambda}}_{+} = (\mathbf{K}^{*}_{+})^{-1}\mathbf{N}^{*}_{+}\dot{\boldsymbol{\varepsilon}} > 0; \quad (\mathbf{N}^{*}_{0} - \mathbf{K}^{*}_{0+}(\mathbf{K}^{*}_{+})^{-1}\mathbf{N}^{*}_{+})\dot{\boldsymbol{\varepsilon}} \leqslant \mathbf{0} \tag{75}$$

and at the same time describing the region where this solution is valid. The response can be represented by

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}\dot{\boldsymbol{\varepsilon}}; \quad \mathbf{D} = \mathbf{D}^{e} + \mathbf{D}^{p}; \quad \mathbf{D}^{p} = -\mathbf{M}^{*}_{+}(\mathbf{K}^{*}_{+})^{-1}\mathbf{N}^{*}_{+} \tag{76}$$

with the tangent stiffness matrix legal in the region described by eqn (75). As in the previous case there are $2^{n} - 1$ such regions and each gives a different form of the plastic stiffness matrix. All matrices involved can be represented by vectors and moduli from different spaces as presented in Section 3.

5.3. *Mixed control*

This case is the most general one and in fact comprises the two previous cases. However, they have been treated separately, because of their practical importance. Assume that the increments of the control and response variables are

$$\dot{\mathbf{c}} = [\dot{\boldsymbol{\sigma}}_1^{\mathrm{T}} \quad \dot{\boldsymbol{\varepsilon}}_2^{\mathrm{T}}]^{\mathrm{T}}; \quad \dot{\mathbf{r}} = [\dot{\boldsymbol{\varepsilon}}_1^{\mathrm{T}} \quad \dot{\boldsymbol{\sigma}}_2^{\mathrm{T}}]^{\mathrm{T}}. \tag{77}$$

Require also that the matrix of generalized plastic moduli $\mathbf{K}$ corresponding to this choice of variables fulfils the uniqueness requirements eqn (57). The elastic response when total unloading occurs is described by

$$\dot{\mathbf{r}} = \mathbf{E}^{\mathrm{e}}\dot{\mathbf{c}} \quad \text{or} \quad \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1 \\ \dot{\boldsymbol{\sigma}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11}^{\mathrm{e}} & \mathbf{E}_{12}^{\mathrm{e}} \\ \mathbf{E}_{21}^{\mathrm{e}} & \mathbf{E}_{22}^{\mathrm{e}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix} \quad \text{if} \quad \mathbf{N}\dot{\mathbf{c}} \leqslant \mathbf{0} \tag{78}$$

The consistency conditions for a given choice of the plastic multiplier vector are

$$\begin{bmatrix} \mathbf{N}_+ \\ \mathbf{N}_0 \end{bmatrix} \dot{\mathbf{c}} - \begin{bmatrix} \mathbf{K}_+ & \mathbf{K}_{+0} \\ \mathbf{K}_{0+} & \mathbf{K}_0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\lambda}}_+ \\ \mathbf{0} \end{bmatrix} \begin{matrix} = \\ \leqslant \end{matrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \mathbf{N}_+\dot{\mathbf{c}} - \mathbf{K}_+\dot{\boldsymbol{\lambda}}_+ = \mathbf{0} \\ \mathbf{N}_0\dot{\mathbf{c}} - \mathbf{K}_{0+}\dot{\boldsymbol{\lambda}}_+ \leqslant \mathbf{0} \end{matrix} \tag{79}$$

and the solution

$$\dot{\boldsymbol{\lambda}}_+ = (\mathbf{K}_+^{-1}\mathbf{N}_+\dot{\mathbf{c}} > 0; \quad (\mathbf{N}_0 - \mathbf{K}_{0+}\mathbf{K}_+^{-1}\mathbf{N}_+)\dot{\mathbf{c}} \leqslant \mathbf{0} \tag{80}$$

describes also the region where it is valid. Splitting matrices $\mathbf{N}$ and $\mathbf{M}$ as follows

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_+ \\ \mathbf{N}_0 \end{bmatrix} = [\mathbf{N}_1, \mathbf{N}_2] = \begin{bmatrix} \mathbf{N}_{+1} & \mathbf{N}_{+2} \\ \mathbf{N}_{01} & \mathbf{N}_{02} \end{bmatrix}; \quad \mathbf{M} = [\mathbf{M}_+, \mathbf{M}_0] = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{1+} & \mathbf{M}_{10} \\ \mathbf{M}_{2+} & \mathbf{M}_{20} \end{bmatrix} \tag{81}$$

and calculating

$$\begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\varepsilon}}_1^{\mathrm{p}} \\ \dot{\boldsymbol{\sigma}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11}^{\mathrm{e}} & \mathbf{E}_{12}^{\mathrm{e}} \\ \mathbf{E}_{21}^{\mathrm{e}} & \mathbf{E}_{22}^{\mathrm{e}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\varepsilon}}_2^{\mathrm{p}} \end{bmatrix}; \quad \begin{matrix} \dot{\boldsymbol{\varepsilon}}_1^{\mathrm{p}} = \mathbf{M}_{+1}^{\#}\dot{\boldsymbol{\lambda}}_+ \\ \dot{\boldsymbol{\varepsilon}}_2^{\mathrm{p}} = \mathbf{M}_{+2}^{\#}\dot{\boldsymbol{\lambda}}_+ \end{matrix}$$

$$\dot{\boldsymbol{\varepsilon}}_1 = \mathbf{E}_{11}^{\mathrm{e}}\dot{\boldsymbol{\sigma}}_1 + \mathbf{E}_{12}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}}_2 + \mathbf{M}_{+1}\mathbf{K}_+^{-1}[\mathbf{N}_{+1} \quad \mathbf{N}_{+2}] \begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix}$$

$$\dot{\boldsymbol{\sigma}}_2 = \mathbf{E}_{21}^{\mathrm{e}}\dot{\boldsymbol{\sigma}}_1 + \mathbf{E}_{22}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}}_2 - \mathbf{M}_{+2}\mathbf{K}_+^{-1}[\mathbf{N}_{+1} \quad \mathbf{N}_{+2}] \begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix} \tag{82}$$

the following constitutive relation is obtained

$$\dot{\mathbf{r}} = \mathbf{E}\dot{\mathbf{c}} \quad \text{or} \quad \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1 \\ \dot{\boldsymbol{\sigma}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\sigma}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix} \tag{83}$$

where

$$\mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}} \quad \text{or} \quad \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11}^{\mathrm{e}} & \mathbf{E}_{12}^{\mathrm{e}} \\ \mathbf{E}_{21}^{\mathrm{e}} & \mathbf{E}_{22}^{\mathrm{e}} \end{bmatrix} + \begin{bmatrix} \mathbf{E}_{11}^{\mathrm{p}} & \mathbf{E}_{12}^{\mathrm{p}} \\ \mathbf{E}_{21}^{\mathrm{p}} & \mathbf{E}_{22}^{\mathrm{p}} \end{bmatrix} \tag{84}$$

The plastic part of the constitutive matrix can be expressed as

$$\mathbf{E}^{p} = \begin{bmatrix} \mathbf{M}_{1+} \\ -\mathbf{M}_{2+} \end{bmatrix} \mathbf{K}_{-}^{-1}[\mathbf{N}_{+1}, \mathbf{N}_{+2}] = \begin{bmatrix} \mathbf{M}_{1+}\mathbf{K}_{+}^{-1}\mathbf{N}_{+1} & \mathbf{M}_{1+}\mathbf{K}_{+}^{-1}\mathbf{N}_{+2} \\ -\mathbf{M}_{2+}\mathbf{K}_{+}^{-1}\mathbf{N}_{+1} & -\mathbf{M}_{2+}\mathbf{K}_{+}^{-1}\mathbf{N}_{+2} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11}^{p} & \mathbf{E}_{12}^{p} \\ \mathbf{E}_{21}^{p} & \mathbf{E}_{22}^{p} \end{bmatrix} \tag{85}$$

and this form is valid within the region eqn (80). Since there are $2^n - 1$ such regions then there are also as many forms of the plastic part of the constitutive matrix eqn (85). All applied matrices can be represented in terms of quantities from different spaces describing the state as indicated in equation (41). In particular if $\dot{\mathbf{c}} = \dot{\boldsymbol{\sigma}}_1 = \dot{\boldsymbol{\sigma}}$ is chosen as the control the pure stress case is obtained and for $\dot{\mathbf{c}} = \dot{\boldsymbol{\varepsilon}}_2 = \dot{\boldsymbol{\varepsilon}}$ the pure strain control case is recovered.

6. SYNTHESIS OF RESPONSE UNIQUENESS

In the previous chapters criteria have been defined to obtain a unique response for stress, strain and mixed control. In each case this criterion is of the same form, but involves different matrices of generalized plastic moduli. It is interesting to know how these criteria relate to each other. However, it seems to be impossible to draw any conclusions in a general case and, therefore, the considered case will be restricted to the associated flow rules, $\mathbf{M}^{\#} = (\mathbf{N}^{\#})^{\mathrm{T}}$, and symmetric matrix of plastic moduli, $\mathbf{H} = \mathbf{H}^{\mathrm{T}}$. Thus the uniqueness condition (57) is equivalent to the requirement that the matrix of generalized plastic moduli $\mathbf{K}$ is positive definite. According to eqn (41) this matrix can be expressed as

$$\mathbf{K} = \mathbf{H} + \mathbf{N}_2^{\#}\mathbf{E}_{22}^{e}(\mathbf{N}_2^{\#})^{\mathrm{T}} = \mathbf{K}^* - \mathbf{N}_1^{*}\mathbf{E}_{11}^{e}(\mathbf{N}_1^{*})^{\mathrm{T}} \tag{86}$$

where matrices $\mathbf{E}_{11}^{e} = (\mathbf{D}_{11}^{e})^{-1}$ and $\mathbf{E}_{22}^{e} = (\mathbf{C}_{22}^{e})^{-1}$ are both positive definite and symmetric. Therefore

$$\forall \mathbf{x} \quad \mathbf{x}^{\mathrm{T}}\mathbf{N}_1^{*}\mathbf{E}_{11}^{e}(\mathbf{N}_1^{*})^{\mathrm{T}}\mathbf{x} \geqslant 0 \quad \mathbf{x}^{\mathrm{T}}\mathbf{N}_2^{\#}\mathbf{E}_{22}^{e}(\mathbf{N}_2^{\#})^{\mathrm{T}}\mathbf{x} \geqslant 0 \tag{87}$$

i.e. matrices $\mathbf{N}_1^{*}\mathbf{E}_{11}^{e}(\mathbf{N}_1^{*})^{\mathrm{T}}$ and $\mathbf{N}_2^{\#}\mathbf{E}_{22}^{e}(\mathbf{N}_2^{\#})^{\mathrm{T}}$ are positive semidefinite and symmetric. Furthermore, considering matrices $\mathbf{K}$ it follows that

$$\begin{aligned} & \mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{N}_2^{\#}\mathbf{E}_{22}^{e}(\mathbf{N}_2^{\#})^{\mathrm{T}}\mathbf{x} \geqslant \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{x} \\ \forall \mathbf{x} \quad & \mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{K}^*\mathbf{x} - \mathbf{x}^{\mathrm{T}}\mathbf{N}_1^{*}\mathbf{E}_{11}^{e}(\mathbf{N}_1^{*})^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}}\mathbf{K}^*\mathbf{x} \\ & \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} \leqslant \mathbf{x}^{\mathrm{T}}\mathbf{K}^*\mathbf{x} \end{aligned} \tag{88}$$

The consequence is that if matrix $\mathbf{H}$ is positive definite then all other matrices $\mathbf{K}$ are also positive definite. Similarly, if $\mathbf{K}^*$ is not positive definite so are the other matrices $\mathbf{K}$, but not the other way around. In other words, if the uniqueness of the response is assured for stress control, then any other control will also give a unique response. Moreover, if the pure strain control gives a nonunique response, then so do all other controls. However, it is necessary to remember that these relations are true only for associated flow rules and symmetric matrices of plastic moduli $\mathbf{H}$.

7. EXAMPLE

To illustrate the general results consider Tresca's criterion with associated flow rules. Let us restrict our considerations to the principal stress and conjugate strain components

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^{\mathrm{T}}; \quad \boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \varepsilon_3]^{\mathrm{T}} \tag{89}$$

The isotropic elastic stiffness and compliance matrices are

$$\mathbf{D}^{\mathrm{e}} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}; \quad \mathbf{C}^{\mathrm{e}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \tag{90}$$

The Tresca yield criteria can be written as

$$F^{\#}_{ij} = \sigma_i - \sigma_j - \kappa_{ij} = 0 \quad i,j = 1,2,3 \quad i \neq j \tag{91}$$

and they define six planes parallel to the hydrostatic axis, $\sigma_1 = \sigma_2 = \sigma_3$ and one of the principal stress axis σ_k, where $i \neq k \neq j$. Notice that the indices indicate which stresses appear in the yield criterion and not the number of the yield surface. The associated flow rules can be written

$$\begin{aligned} & \mathbf{n}^{\#}_{12} = [1 \quad -1 \quad 0]^{\mathrm{T}}; \quad \mathbf{n}^{\#}_{21} = [-1 \quad 1 \quad 0]^{\mathrm{T}} \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{ij} = \dot{\lambda}_{ij}\mathbf{n}^{\#}_{ij}, \quad & \mathbf{n}^{\#}_{23} = [0 \quad 1 \quad -1]^{\mathrm{T}}; \quad \mathbf{n}^{\#}_{32} = [0 \quad -1 \quad 1]^{\mathrm{T}} \\ & \mathbf{n}^{\#}_{31} = [-1 \quad 0 \quad 1]^{\mathrm{T}}; \quad \mathbf{n}^{\#}_{13} = [1 \quad 0 \quad -1]^{\mathrm{T}} \end{aligned} \tag{92}$$

Consider two extreme cases of linear hardening: independent when hardening of one yield surface does not influence the other yield surfaces, and isotropic when all surfaces exhibit exactly the same hardening. In the first case hardening of a given yield surface depends on plastic strains generated by this surface only whereas in the second case hardening is identical for all surfaces and depends on the total length of the plastic strain increment, i.e.

$$\dot{\kappa}^{\mathrm{ind}}_{ij} = h(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{ij}) = h\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{ij}\|; \quad \dot{\kappa}^{\mathrm{iso}}_{ij} = h(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) = h\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\| \tag{93}$$

where $\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}_{ij}\|$ and $\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\|$ denote the norms of the plastic strain increment generated by the yield surface $F^{\#}_{ij} = 0$ and of the total plastic strain increment, respectively. Furthermore, h is assumed to be the same constant for all surfaces. Since the Euclidean norm is the same for all normal vectors, $\|\mathbf{n}^{\#}_{kl}\| = \sqrt{2}$, the matrices of plastic moduli are

$$\mathbf{H}^{\mathrm{ind}} = H\mathbf{I}; \quad \mathbf{H}^{\mathrm{iso}} = H\mathbf{1} \tag{94}$$

where $H = h\sqrt{2}$, $\mathbf{I}$ is the unit matrix and $\mathbf{1}$ is the matrix with all entries equal to one. Notice that the first matrix satisfies the uniqueness condition as long as $H > 0$, whereas the second matrix leads to zero minors when its size is larger than 1. Therefore, the Tresca criterion with isotropic hardening gives a non-unique response under stress control and only the independent hardening will be discussed further for this control.

Considering, for example, the response at the corner between surfaces $F^{\#}_{12} = 0$ and $F^{\#}_{13} = 0$ the matrix of normal vectors and the matrix of plastic moduli are

$$\mathbf{N}^{\#} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}; \quad \mathbf{H}^{\mathrm{ind}} = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \tag{95}$$

The unloading occurs when

$$\mathbf{N}^{\#}\dot{\boldsymbol{\sigma}} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\dot{\boldsymbol{\sigma}} \leqslant \mathbf{0} \quad \text{or} \quad \begin{aligned} \dot{\sigma}_1 - \dot{\sigma}_2 \leqslant 0 \\ \dot{\sigma}_1 - \dot{\sigma}_3 \leqslant 0 \end{aligned} \tag{96}$$

Loading of the surface $F^{\#}_{12} = 0$ while unloading $F^{\#}_{13} = 0$ leads to the decomposition

$$\mathbf{N}_+^\# = [1 \quad -1 \quad 0]; \quad \mathbf{H}_+ = [H]; \quad \mathbf{H}_{+0} = [0]$$
$$\mathbf{N}_0^\# = [1 \quad 0 \quad -1]; \quad \mathbf{H}_{0+} = [0]; \quad \mathbf{H}_0 = [H] \tag{97}$$

and the region where it occurs is defined by

$$\mathbf{H}_+^{-1}\mathbf{N}_+^\#\dot{\boldsymbol{\sigma}} = \frac{1}{H}[1 \quad -1 \quad 0]\dot{\boldsymbol{\sigma}} > 0; \quad (\mathbf{N}_0^\# - \mathbf{H}_{0+}\mathbf{H}_+^{-1}\mathbf{N}_+^\#)\dot{\boldsymbol{\sigma}} = [1 \quad 0 \quad -1]\dot{\boldsymbol{\sigma}} \leqslant 0$$
$$\text{or} \quad \dot{\sigma}_1 - \dot{\sigma}_2 > 0 \quad \dot{\sigma}_1 - \dot{\sigma}_3 \leqslant 0 \tag{98}$$

since $H > 0$. The plastic compliance matrix can be established as

$$\mathbf{C}^{\mathrm{p}}_{(12)} = (\mathbf{N}_+^\#)^{\mathrm{T}}\mathbf{H}_+^{-1}\mathbf{N}_+^\# = \frac{1}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{99}$$

Likewise loading of the second yield surface while unloading of the first one gives

$$\mathbf{C}^{\mathrm{p}}_{(13)} = \frac{1}{H}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{if} \quad \begin{matrix} \dot{\sigma}_1 - \dot{\sigma}_3 > 0 \\ \dot{\sigma}_1 - \dot{\sigma}_2 \leqslant 0 \end{matrix} \tag{100}$$

Notice that due to the symmetry the same result can be obtained just by exchanging the last two rows and columns in eqn (99).

Loading of both surfaces leads to the following region and plastic compliance matrix

$$\mathbf{H}^{-1}\mathbf{N}^\#\dot{\boldsymbol{\sigma}} = \frac{1}{H}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\dot{\boldsymbol{\sigma}} > \mathbf{0} \quad \text{or} \quad \begin{matrix} \dot{\sigma}_1 - \dot{\sigma}_3 > 0 \\ \dot{\sigma}_1 - \dot{\sigma}_2 > 0 \end{matrix}$$

$$\mathbf{C}^{\mathrm{p}} = (\mathbf{N}^\#)^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{N}^\# = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}^{-1}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$
$$= \frac{1}{H}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{101}$$

Summarizing

$$\mathbf{C}^{\mathrm{p}} = \begin{cases} \mathbf{0} & \text{if} \quad \dot{\sigma}_1 - \dot{\sigma}_2 \leqslant 0 \quad \dot{\sigma}_1 - \dot{\sigma}_3 \leqslant 0 \\ \mathbf{C}^{\mathrm{p}}_{(12)} & \text{if} \quad \dot{\sigma}_1 - \dot{\sigma}_2 > 0 \quad \dot{\sigma}_1 - \dot{\sigma}_3 \leqslant 0 \\ \mathbf{C}^{\mathrm{p}}_{(13)} & \text{if} \quad \dot{\sigma}_1 - \dot{\sigma}_3 > 0 \quad \dot{\sigma}_1 - \dot{\sigma}_2 \leqslant 0 \\ \mathbf{C}^{\mathrm{p}}_{(12)} + \mathbf{C}^{\mathrm{p}}_{(13)} & \text{if} \quad \dot{\sigma}_1 - \dot{\sigma}_2 > 0 \quad \dot{\sigma}_1 - \dot{\sigma}_3 > 0 \end{cases} \tag{102}$$

where $\mathbf{C}^{\mathrm{p}}_{(12)}$ and $\mathbf{C}^{\mathrm{p}}_{(13)}$ are given by eqns (99) and (100). So independent hardening results in fully independent plastic compliance matrices.

Choose the control and response as

$$\mathbf{c} = [\varepsilon_1, \sigma_2, \sigma_3]^{\mathrm{T}} \quad \mathbf{r} = [\sigma_1, \varepsilon_2, \varepsilon_3]^{\mathrm{T}} \tag{103}$$

The partially inverted elastic relationship is

$$\begin{bmatrix} \sigma_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} E & \nu & \nu \\ -\nu & (1-\nu^2)/E & -\nu(1+\nu)/E \\ -\nu & -\nu(1+\nu)/E & (1-\nu^2)/E \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{104}$$

Considering the same vertex as before the relevant submatrices are

$$\mathbf{N}_1^{\#} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \mathbf{N}_2^{\#} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{E}_{11}^{\mathrm{e}} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu \\ -\nu & 1-\nu \end{bmatrix};$$

$$\mathbf{E}_{12}^{\mathrm{e}} = \begin{bmatrix} -\nu \\ -\nu \end{bmatrix}; \quad \mathbf{E}_{21}^{\mathrm{e}} = [\nu, \nu]; \quad \mathbf{E}_{22}^{\mathrm{e}} = [E] \tag{105}$$

and the matrix of plastic moduli is modified to

$$\mathbf{K} = \mathbf{H} + \mathbf{N}_2^{\#} \mathbf{E}_{22}^{\mathrm{e}} (\mathbf{N}_2^{\#})^{\mathrm{T}} = \mathbf{H} + E\mathbf{1} \tag{106}$$

For independent and isotropic hardening these matrices look as follows

$$\mathbf{K}^{\mathrm{ind}} = \begin{bmatrix} H+E & E \\ E & H+E \end{bmatrix}; \quad \mathbf{K}^{\mathrm{iso}} = \begin{bmatrix} H+E & H+E \\ H+E & H+E \end{bmatrix}$$

$$det(\mathbf{K}^{\mathrm{ind}}) = (H+E)^2 - E^2; \quad \det(\mathbf{K}^{\mathrm{iso}}) = 0 \tag{107}$$

i.e. the independent hardening gives a unique response as long as $H > 0$, whereas the isotropic hardening is still non-unique in spite of the partial strain control. Once again only independent hardening will be considered for this control.

The matrix of normal vectors is modified according to

$$\mathbf{N}_1 = \mathbf{N}_1^{\#} + \mathbf{N}_2^{\#} \mathbf{E}_{21}^{\mathrm{e}} = \begin{bmatrix} -(1-\nu) & \nu \\ \nu & -(1-\nu) \end{bmatrix};$$

$$\mathbf{N}_2 = \mathbf{N}_2^{\#} \mathbf{E}_{22}^{\mathrm{e}} = \begin{bmatrix} E \\ E \end{bmatrix};$$

$$\mathbf{N} = \begin{bmatrix} E & -(1-\nu) & \nu \\ E & \nu & -(1-\nu) \end{bmatrix} \tag{108}$$

where the original order is restored in $\mathbf{N}$, i.e. $\mathbf{N}$ is written as $[\mathbf{N}_2, \mathbf{N}_1]$ to be consistent with $\dot{\mathbf{c}}$. Unloading occurs when

$$\mathbf{N}\dot{\mathbf{c}} = \begin{bmatrix} E & -(1-\nu) & \nu \\ E & \nu & -(1-\nu) \end{bmatrix} \dot{\mathbf{c}} \leqslant \mathbf{0} \tag{109}$$

Loading the first surface while unloading the second gives the following region of occurrence

$$\mathbf{K}_{+}^{-1} \mathbf{N}_{+} \dot{\mathbf{c}} = \frac{1}{H+E} [E, -(1-\nu), \nu] \dot{\mathbf{c}} > 0$$

$$(\mathbf{N}_0 - \mathbf{K}_{0+} \mathbf{K}_{+}^{-1} \mathbf{N}_{+}) \dot{\mathbf{c}} = \frac{1}{H+E} [HE, \nu H + E, -[(1-\nu)H + E]] \dot{\mathbf{c}} \leqslant 0 \tag{110}$$

where the multiplier $1/(H+E)$ can be neglected as $H > 0$ and $E > 0$.

The plastic constitutive matrix is established from eqn (85) giving

$$\mathbf{E}^{p}_{(12)} = \frac{1}{H+E}\begin{bmatrix} -E^2 & (1-\nu)E & -\nu E \\ -(1-\nu)E & (1-\nu)^2 & -\nu(1-\nu) \\ \nu E & -\nu(1-\nu) & \nu^2 \end{bmatrix} \tag{111}$$

The subindex indicates that this matrix holds for loading of the yield surface $F^{\#}_{12} = 0$. Similar considerations for loading of the second yield surface while unloading of the first one or just exchanging the last two rows and columns give

$$\mathbf{E}^{p}_{(13)} = \frac{1}{H+E}\begin{bmatrix} -E^2 & -\nu E & (1-\nu)E \\ \nu E & \nu^2 & -\nu(1-\nu) \\ -(1-\nu)E & -\nu(1-\nu) & (1-\nu)^2 \end{bmatrix} \tag{112}$$

if

$$[E, \nu, -(1-\nu)]\dot{\mathbf{c}} > 0; \quad [HE, -[(1-\nu)H+E], \nu H+E]\dot{\mathbf{c}} \leqslant 0$$

Loading of both surfaces results in the conditions

$$\mathbf{K}^{-1}\mathbf{N}\dot{\mathbf{c}} = \frac{1}{H+2E}\begin{bmatrix} E & \nu-1-E/H & \nu+E/H \\ E & \nu+E/H & \nu-1-E/H \end{bmatrix}\dot{\mathbf{c}} > \mathbf{0} \tag{113}$$

where once again the multiplier $1/(H+2E) > 0$ can be neglected and

$$\mathbf{E}^{p} = \frac{1}{H+2E}\begin{bmatrix} -2E^2 & E(1-2\nu) & E(1-2\nu) \\ -E(1-2\nu) & -2\nu(1-\nu)+1+E/H & 2\nu(\nu-1)-E/H \\ -E(1-2\nu) & 2\nu(\nu-1)-E/H & -2\nu(1-\nu)+1+E/H \end{bmatrix} \tag{114}$$

Notice that in this case the above plastic constitutive matrix cannot be represented as a sum of $\mathbf{E}^{p}_{(12)}$ and $\mathbf{E}^{p}_{(13)}$.

Let us exchange the control and response variables in eqn (103), i.e. right now

$$\mathbf{c} = [\sigma_1, \varepsilon_2, \varepsilon_3]^{\mathrm{T}}; \quad \mathbf{r} = [\varepsilon_1, \sigma_2, \sigma_3]^{\mathrm{T}} \tag{115}$$

and the elastic relationship eqn (104) can simply be inverted leading to

$$\begin{bmatrix} \varepsilon_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{1-\nu}\begin{bmatrix} (1+\nu)(1-2\nu)/E & -\nu & -\nu \\ \nu & E/(1+\nu) & E\nu/1+\nu \\ \nu & E\nu/1+\nu & E/(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{116}$$

Considering the same corner as before the matrix of generalized plastic moduli turns out to be

$$\mathbf{K} = \mathbf{H}+\mathbf{N}^{\#}_2\mathbf{E}^{e}_{22}(\mathbf{N}^{\#}_2)^{\mathrm{T}} = \mathbf{H}+\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} = \mathbf{H}+E'\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \tag{117}$$

where $E' = E/(1-\nu^2)$. For independent and isotropic hardening one obtains

$$\mathbf{K}^{\text{ind}} = \begin{bmatrix} H+E' & \nu E' \\ \nu E' & H+E' \end{bmatrix}; \quad \mathbf{K}^{\text{iso}} = \begin{bmatrix} H+E' & H+\nu E' \\ H+\nu E' & H+E' \end{bmatrix}$$

$$det(\mathbf{K}^{\text{ind}}) = (H+E')^2-(\nu E')^2; \quad \det(\mathbf{K}^{\text{iso}}) = (H+E')^2-(H+\nu E')^2 \tag{118}$$

The uniqueness conditions require that $H > -(1-\nu)E' = -E/(1+\nu)$ for independent hardening and $H > -(1+\nu)E'/2 = -(1-\nu)E/2$ for isotropic hardening, i.e. the latter case can also give a unique response. The matrix of normal vectors is modified to

$$\mathbf{N} = \begin{bmatrix} \nu' & -E' & -E'\nu \\ \nu' & -E'\nu & -E' \end{bmatrix}; \quad \nu' = (1-2\nu)/(1-\nu) \tag{119}$$

Unloading occurs when

$$\mathbf{N}\dot{\mathbf{c}} = \begin{bmatrix} \nu' & -E' & -E'\nu \\ \nu' & -E'\nu & -E' \end{bmatrix}\dot{\mathbf{c}} \leqslant \mathbf{0} \tag{120}$$

Loading of the surface $F^{\#}_{12} = 0$ while unloading $F^{\#}_{13} = 0$ occurs in the region defined by

$$\mathbf{K}_{+}^{-1}\mathbf{N}_{+}\dot{\mathbf{c}} = \frac{1}{H+E'}[\nu' \quad -E' \quad -\nu E']\dot{\mathbf{c}} > 0$$

$$(\mathbf{N}_0 - \mathbf{K}_{0+}^{\text{ind}}\mathbf{K}_{+}^{-1}\mathbf{N}_{+})\dot{\mathbf{c}} = \frac{E'}{H+E'}\left[\nu'\frac{H}{E'}+1-2\nu,\ -\nu H,\ -H-(1-\nu^2)E'\right]\dot{\mathbf{c}} \leqslant 0$$

$$(\mathbf{N}_0 - \mathbf{K}_{0+}^{\text{iso}}\mathbf{K}_{+}^{-1}\mathbf{N}_{+})\dot{\mathbf{c}} = \frac{(1-\nu)E'}{H+E'}[\nu', H', -H-(1+\nu)E']\dot{\mathbf{c}} \leqslant 0 \tag{121}$$

where the second condition is different for independent and isotropic hardening. However, the plastic constitutive matrix is the same and can be established as

$$\mathbf{E}^{\mathrm{p}} = \frac{1}{H+E'}\begin{bmatrix} \nu'^2 & -\nu'E' & -\nu\nu'E' \\ \nu'E' & -E'^2 & -\nu(E')^2 \\ \nu\nu'E' & -\nu(E')^2 & -(\nu E')^2 \end{bmatrix} \tag{122}$$

Similar considerations can also be performed for loading of the second yield surface while unloading the first, but it is sufficient to exchange two rows and columns as shown earlier.

Loading of both surfaces results in the condition

$$\mathbf{K}_{\text{ind}}^{-1}\mathbf{N}\dot{\mathbf{c}} = \frac{E'}{(H+E')^2-(\nu E')^2} \times \begin{bmatrix} \nu'\left(\frac{H}{E'}+1-\nu\right) & -[H+(1-\nu^2)E'] & -\nu H \\ \nu'\left(\frac{H}{E'}+1-\nu\right) & -\nu H & -[H+(1-\nu^2)E'] \end{bmatrix}\dot{\mathbf{c}} > \mathbf{0}$$

$$\mathbf{K}_{\text{iso}}^{-1}\mathbf{N}\dot{\mathbf{c}} = \frac{1}{2H+(1+\nu)E'}\begin{bmatrix} \nu' & -(H+(1-\nu)E') & H \\ \nu' & H & -(H+(1+\nu)E') \end{bmatrix}\dot{\mathbf{c}} > \mathbf{0} \tag{123}$$

and plastic constitutive matrices are different for each hardening model

$$\mathbf{E}^{\mathrm{p}}_{\mathrm{ind}} = \frac{E'}{(H+E')^2-(\nu E')^2}$$

$$\times \begin{bmatrix} 2\nu'^2\left(1-\nu+\dfrac{H}{E'}\right) & -\nu'(1+\nu)[H+(1-\nu)E'] & -\nu'(1+\nu)[H+(1-\nu)E'] \\ \nu'(1+\nu)[H+(1-\nu)E'] & -[(1+\nu^2)H+(1-\nu^2)E']E' & -\nu(2H+(1-\nu^2)E')E' \\ \nu'(1+\nu)[H+(1-\nu)E'] & -\nu(2H+(1-\nu^2)E')E' & -[(1+\nu^2)H+(1-\nu^2)E']E' \end{bmatrix}$$

$$\mathbf{E}^{\mathrm{p}}_{\mathrm{iso}} = \frac{1}{2H+(1+\nu)E'}$$

$$\times \begin{bmatrix} 2\nu'^2 & -\nu'(1+\nu)E' & -\nu'(1+\nu)E' \\ \nu'(1+\nu)E' & -[(1+\nu)E'+(1-\nu)H]E' & -[\nu(1+\nu)E'-(1-\nu)H]E' \\ \nu'(1+\nu)E' & -[\nu(1+\nu)E'-(1-\nu)H]E' & -[(1+\nu)E'+(1-\nu)H]E' \end{bmatrix}. \quad (124)$$

Finally, consider a pure strain control. The matrix of plastic moduli under strain control becomes

$$\mathbf{K}^* = \mathbf{H}+\mathbf{N}^{\#}\mathbf{D}^{\mathrm{e}}(\mathbf{N}^{\#})^{\mathrm{T}} = \mathbf{H}+\frac{E}{1+\nu}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (125)$$

and denoting $E'' = E/(1+\nu)$ the matrices for independent and isotropic hardening can be written

$$\mathbf{K}^*_{\mathrm{ind}} = \begin{bmatrix} H+2E'' & E'' \\ E'' & H+2E'' \end{bmatrix}; \quad \mathbf{K}^*_{\mathrm{iso}} = \begin{bmatrix} H+2E'' & H+E'' \\ H+E'' & H+2E'' \end{bmatrix}$$

$$det(\mathbf{K}^*_{\mathrm{ind}}) = (H+2E'')^2-(E'')^2; \quad \det(\mathbf{K}^*_{\mathrm{iso}}) = (H+2E'')^2-(H+E'')^2 \quad (126)$$

The uniqueness condition for independent hardening is satisfied if $H > -E'' = -E/(1+\nu)$ and for isotropic hardening if $H > -\frac{3}{2}E'' = -3E/[2(1+\nu)]$. Notice that for independent hardening it is the same requirement as in the previous case, whereas for isotropic hardening the current condition is less severe than the previous one.

The matrix of normal vectors is transformed to the strain space as follows

$$\mathbf{N}^* = \mathbf{N}^{\#}\mathbf{D}^{\mathrm{e}} = \frac{E}{1+\nu}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = E''\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad (127)$$

Unloading occurs when

$$\mathbf{N}^*\dot{\boldsymbol{\varepsilon}} = E''\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\dot{\boldsymbol{\varepsilon}} \leqslant \mathbf{0} \quad \text{or} \quad \begin{matrix} \dot{\varepsilon}_1-\dot{\varepsilon}_2 \leqslant 0 \\ \dot{\varepsilon}_1-\dot{\varepsilon}_3 \leqslant 0 \end{matrix} \quad (128)$$

Loading of just the first surface $F^{\#}_{12} = 0$ leads to the region defined by

$$(\mathbf{K}^*_+)^{-1}\mathbf{N}^*_+\dot{\boldsymbol{\varepsilon}} = \frac{E''}{H+2E''}[1 \quad -1 \quad 0]\dot{\boldsymbol{\varepsilon}} > 0$$

$$(\mathbf{N}^*_0-\mathbf{K}^{*\mathrm{ind}}_{0+}(\mathbf{K}^*_+)^{-1}\mathbf{N}^*_+)\dot{\boldsymbol{\varepsilon}} = \frac{E''}{H+2E''}[H+E'', E'', -H-2E'']\dot{\boldsymbol{\varepsilon}} \leqslant 0$$

$$(\mathbf{N}^*_0-\mathbf{K}^{*\mathrm{iso}}_{0+}(\mathbf{K}^*_+)^{-1}\mathbf{N}^*_+)\dot{\boldsymbol{\varepsilon}} = \frac{E''}{H+2E''}[E'', H+E'', -H-2E'']\dot{\boldsymbol{\varepsilon}} \leqslant 0 \quad (129)$$

where, as before, the second condition depends on the type of hardening. The plastic stiffness matrix can be established as

$$\mathbf{D}^{p} = -(\mathbf{N}^{*}_{\pm})^{T}(\mathbf{K}^{*}_{+})^{-1}\mathbf{N}^{*}_{+} = -\frac{E''^{2}}{H+2E''}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{130}$$

Loading of both surfaces results in the conditions

$$(\mathbf{K}^{*}_{\text{ind}})^{-1}\mathbf{N}^{*}\dot{\boldsymbol{\varepsilon}} = \frac{E''}{(H+E'')(H+3E'')}\begin{bmatrix} H+E'' & -(H+2E'') & E'' \\ H+E'' & E'' & -(H+2E'') \end{bmatrix}\dot{\boldsymbol{\varepsilon}} > \mathbf{0}$$

$$(\mathbf{K}^{*}_{\text{iso}})^{-1}\mathbf{N}^{*}\dot{\boldsymbol{\varepsilon}} = \frac{1}{2H+3E''}\begin{bmatrix} E'' & -(H+2E'') & H+E'' \\ E'' & H+E'' & -(H+2E'') \end{bmatrix}\dot{\boldsymbol{\varepsilon}} > \mathbf{0} \tag{131}$$

and

$$\mathbf{D}^{p}_{\text{ind}} = \frac{E''^{2}}{(H+E'')(H+3E'')}\begin{bmatrix} -2(H+E'') & H+E'' & H+E'' \\ H+E'' & -(H+2E'') & E'' \\ H+E'' & E'' & -(H+2E'') \end{bmatrix}$$

$$\mathbf{D}^{p}_{\text{iso}} = \frac{E''}{2H+3E''}\begin{bmatrix} -2E'' & E'' & E'' \\ E'' & -(H+2E'') & H+E'' \\ E'' & H+E'' & -(H+2E'') \end{bmatrix} \tag{132}$$

All other corners and control can be described in a similar way, but it seems unnecessary to elaborate any further. Notice that for independent hardening the special case described in Section 4.1.2. can also occur.

8. CONCLUSIONS

The developed structure allows us to handle far more complicated cases with as many yield surfaces as required. However, from a practical point of view the most important task is to be able to secure uniqueness of the response. As shown in the above example it is not so easy to achieve this even for the Tresca criterion with isotropic hardening. Neither the stress control nor the first considered mixed control did lead to a unique response. However, if calculations are made locally under strain control and the global equilibrium is achieved via iterations, such ambiguities may escape our attention. Therefore, local criteria for different kinds of control are of great importance. For example, if optimization is used to identify material parameters the direct application of mixed control allows us to check the uniqueness of response ensuring the correct behaviour of a constitutive model. Furthermore, due to mixed control, unnecessary iterations are avoided, significantly speeding up the identification process.

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REFERENCES

Drucker, D. C. (1959). A definition of a stable inelastic material. *Journal of Applied Mechanics*, **26**, 101.
Hill, R. (1983). On intrinsic eigenstates in plasticity with generalized variables. *Mathematical Proceedings of the Cambridge Philosophical Society*, **93**, 177.

Klisinski, M., Mroz, Z. and Runesson, K. (1992). Structure of constitutive equations in plasticity for different choices of state and control variables. *International Journal of Plasticity*, **8,** 221–243.

Koiter, W. T. (1953). Stress–strain relations, uniqueness and variational theorems for elastic–plastic materials with singular yield surface. *Quarterly Applied Mathematics*, **11,** 350–354.

Koiter, W. T. (1960). General theorems for elastic–plastic solids. *Progress in Solid Mechanics*, Vol. 1, pp. 165–221. North-Holland, Amsterdam.

Lubliner, J. (1987). On uniqueness of solutions in generalized plasticity at infinitesimal deformation. *International Journal of Solids and Structures*, **23,** 261–266.

Maier, G. and Hueckel, T. (1979). Non-associated and coupled flow rules of elastoplasticity for rock-like materials. *International Journal of Rock Mechanics and Minimum Science Geometry Abstracts*, **16,** 77–92.

Mroz, Z. (1963). On forms of constitutive laws for elastic–plastic solids. *Archiwum Mechaniki Stosowanej*, **18,** 1.

Mroz, Z. (1966). Non-associated flow laws in plasticity. *Journal Méchanique*, **2,** 21–42.

Runesson, K., Axelsson, K. and Klisinski, M. (1991). Characteristics of constitutive relations in soil plasticity for undrained behaviour. *International Journal of Solids and Structures*, **29,** 363–380.

Sewell, M. J. (1974). Plastic flow rule at a yield vertex. *Journal of the Mechanical Physics of Solids*, **22,** 469–490.

APPENDIX

Theorem

If a square matrix is positive definite then its determinant is positive.

Proof

Any square matrix $\mathbf{A}$ can be decomposed into symmetric and asymmetric parts

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_a \qquad \begin{matrix} \mathbf{A}_s = \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) & \mathbf{A}_s^T = \mathbf{A}_s \\ \mathbf{A}_a = \frac{1}{2}(\mathbf{A}-\mathbf{A}^T) & \mathbf{A}_a^T = -\mathbf{A}_a \end{matrix}$$

Notice that the quadratic form of any square matrix $\mathbf{A}$ is the same as the quadratic form of its transpose and of the symmetric part

$$\mathbf{x}^T\mathbf{A}^T\mathbf{x} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^T = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\mathbf{x}^T\mathbf{A}_s\mathbf{x} = \tfrac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

whereas for the asymmetric part of the quadratic form is always zero

$$\mathbf{x}^T\mathbf{A}_a\mathbf{x} = \tfrac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \tfrac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{x} = 0$$

For a positive definite symmetric matrix $\mathbf{A}_s$ there exist at orthogonal matrix $\mathbf{P}(\mathbf{P}^{-1} = \mathbf{P}^T)$ such that

$$\mathbf{A}_s = \mathbf{PDP}^T \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix} \quad \lambda_i > 0 \quad i = 1, 2, \dots, n$$

where $\mathbf{D}$ is a diagonal matrix containing positive eigenvalues λ_i.

Construct the following transformation

$$\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{P}^T; \quad \mathbf{R}^T = \mathbf{PD}^{-1/2}$$

and apply to matrix $\mathbf{A}$

$$\mathbf{RAR}^T = \mathbf{R}(\mathbf{A}_s + \mathbf{A}_a)\mathbf{R}^T = \mathbf{D}^{-1/2}\mathbf{P}^T\mathbf{PDP}^T\mathbf{PD}^{-1/2} + \mathbf{RA}_a\mathbf{R}^T = \mathbf{I} + \mathbf{RA}_a\mathbf{R}^T$$

Since

$$(\mathbf{RA}_a\mathbf{R}^T)^T = \mathbf{RA}_a^T\mathbf{R}^T = -\mathbf{RA}_a\mathbf{R}^T$$

the matrix $\mathbf{B}_a = \mathbf{RA}_a\mathbf{R}^T$ is asymmetric and as such it cannot have any real eigenvalues except for zero as shown below

$$\mathbf{B}_a\mathbf{x} = \beta\mathbf{x}\forall\mathbf{x} \neq \mathbf{0} \quad 0 = \mathbf{x}^T\mathbf{B}_a\mathbf{x} = \mathbf{x}^T\beta\mathbf{x} \Rightarrow \beta = 0$$

Consider the following polynomial

$$p(\alpha) = \det(\mathbf{I} + \alpha\mathbf{B}_a)$$

and notice that $p(\alpha) = 0$ is equivalent to the requirement that the following system

$$(\mathbf{I}+\alpha\mathbf{B}_a)\mathbf{x} = \mathbf{0}$$

has non-trivial solutions, but since

$$\alpha \neq 0 \quad (\mathbf{I}+\alpha\mathbf{B}_a)\mathbf{x} = \mathbf{0} \Leftrightarrow \mathbf{B}_a\mathbf{x} = -\frac{1}{\alpha}\mathbf{x}$$

such solutions do not exist. Noting that $p(0) = \det(\mathbf{I}) = 1$ it is clear that $p(\alpha)$ must be positive, because it does not have real roots, and in particular

$$p(1) = \det(\mathbf{I}+\mathbf{B}_a) = \det(\mathbf{RAR}^T) = \det(\mathbf{RR}^T)\cdot\det(\mathbf{A}) > 0$$

Moreover

$$\det(\mathbf{RR}^T) = \det(\mathbf{D}^{-1/2}\mathbf{P}^T\mathbf{P}\mathbf{D}^{-1/2}) = \det(\mathbf{D}^{-1}) = \frac{1}{\det(\mathbf{D})} > 0$$

so it has been proved that

$$\det(\mathbf{A}) > 0$$

Inversion in terms of submatrices

Let us consider inversion of the following matrix

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix} \qquad \mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix}$$

Since

$$\mathbf{AA}^{-1} = \mathbf{I} \Rightarrow \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}$$

so four matrix equations are obtained

$$\mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} = \mathbf{I} \qquad \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} = \mathbf{0}$$

$$\mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} = \mathbf{0} \qquad \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} = \mathbf{I}$$

From the second column

$$\mathbf{B}_{12} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}_{22}(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})\mathbf{B}_{22} = \mathbf{I} \qquad \mathbf{B}_{22} = (\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}$$

and from the first column

$$\mathbf{B}_{11} = \mathbf{A}_{11}^{-1}(\mathbf{I}-\mathbf{A}_{12}\mathbf{B}_{21})\mathbf{A}_{21}\mathbf{A}_{11}^{-1}+(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})\mathbf{B}_{21} = \mathbf{0} \qquad \mathbf{B}_{21} = -\mathbf{B}_{22}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$$

Summarizing

$$\mathbf{B}_{22} = (\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1} \qquad \mathbf{B}_{21} = -\mathbf{B}_{22}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$$

$$\mathbf{B}_{12} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}_{22} \qquad \mathbf{B}_{11} = \mathbf{A}_{11}^{-1}(\mathbf{I}-\mathbf{A}_{12}\mathbf{B}_{21})$$

Determinant

Applying cofactor expansion along the last row one gets

$$\det\begin{bmatrix}\mathbf{K}_+ & \mathbf{k}_{+i}\\ \mathbf{k}_{i+} & K_{ii}\end{bmatrix} = K_{i1}C_{i1}+K_{i2}C_{i2}+\cdots+K_{i(i-1)}C_{i(i-1)}+K_{ii}\det(\mathbf{K}_+) = \mathbf{k}_{i+}\begin{bmatrix}C_{i1}\\ C_{i2}\\ \vdots\\ C_{i(i-1)}\end{bmatrix}+K_{ii}\det(\mathbf{K}_+)$$

where C_{ij} are the appropriate cofactors. Calculating

$$\operatorname{adj}(\mathbf{K}_+)\mathbf{k}_{+i} = \begin{bmatrix} C^+_{11} & C^+_{21} & \dots & C^+_{(i-1)1} \\ C^+_{12} & C^+_{22} & \dots & C^+_{(i-1)2} \\ \vdots & \vdots & & \vdots \\ C^+_{1(i-1)} & C^+_{2(i-1)} & \dots & C^+_{(i-1)(i-1)} \end{bmatrix} \begin{bmatrix} K_{1i} \\ K_{2i} \\ \vdots \\ K_{(i-1)i} \end{bmatrix}$$

$$= \begin{bmatrix} C^+_{11}K_{1i} + C^+_{21}K_{2i} & + & + & C^+_{(i-1)1}K_{(i-1)i} \\ C^+_{12}K_{1i} + C^+_{22}K_{2i} & + & + & C^+_{(i-1)2}K_{(i-1)i} \\ & \vdots & & \\ C^+_{1(i-1)} + C^+_{2(i-1)}K_{2i} & + & + & C^+_{(i-1)(i-1)}K_{(i-1)i} \end{bmatrix}$$

it should be noticed that each entry into the vector represents a cofactor expansion along the last column. However, due to the fact that the number of columns has increased by one the sign of cofactors must be changed. Therefore,

$$\begin{bmatrix} C_{i1} \\ C_{i2} \\ \vdots \\ C_{i(i-1)} \end{bmatrix} = -\operatorname{adj}(\mathbf{K}_+)\mathbf{k}_{+1}$$

represents the appropriate cofactors for calculations of the determinant, i.e.

$$\det \begin{bmatrix} \mathbf{K}_+ & \mathbf{k}_{+i} \\ \mathbf{k}_{i+} & K_{ii} \end{bmatrix} = K_{ii} \det(\mathbf{K}_+) - \mathbf{k}_{i+} \operatorname{adj}(\mathbf{K}_+)\mathbf{k}_{+i}$$